Intergrerend project dynamische systemen

3 April 2013

university of groningen

Exam Problem Sheet

The exam consists of 4 problems. You may answer in Dutch or in English. You have 90 minutes to answer the questions. Give brief but precise answers. You can achieve 50 points in total.

1. [**5+5+5 Points.**]

   For each of the following bifurcations of equilibrium points of time continuous systems, give an explicit example (i.e. a one-parameter family of systems showing the respective bifurcation), plot the bifurcation diagram and describe in words the bifurcation scenario.

   (a) Transcritical bifurcation.
   (b) Pitchfork bifurcation.
   (c) Hopf bifurcation.

2. [**6+2+2 Points.**]

   Consider the planar systems

$$X' = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} X \text{ and } Y' = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} Y \,.$$

   (a) Determine the phase portraits of the two systems.
   (b) Determine the canonical forms of the two systems.
   (c) Are the two systems topologically conjugate? Justify your answer.

3. [**2+3+4+4+2 Points.**]

   Consider the damped harmonic oscillator

$$x'' = -x - \nu x' \tag{1}$$

   with $\nu \geq 0$.

   (a) Show that the damped harmonic oscillator has exactly one equilibrium point.
   (b) Show that for $\nu = 0$, the energy

$$E = \frac{1}{2}x'^2 + \frac{1}{2}x^2$$

   is conserved along solution curves.

(c) Show that for $\nu = 0$, the equilibrium point is stable but not asymptotically stable.

(d) Use the Lasalle Invariance Principle to show that for $\nu > 0$, the equilibrium point is asymptotically stable and determine the basin of attraction.

(e) What can you generally say about the stability of the equilibrium of Equation (1) if a term $f(x)$ is added to the right hand side of Equation (1) when f is a polynomial in x with lowest order term x^2? Distinguish between $\nu = 0$ and $\nu > 0$.

4. **[5+5 Points.]**

Consider the discrete time system $x_{n+1} = f_\lambda(x_n)$ where $\lambda \in \mathbb{R}$ is a parameter. Prove that if the system has a fixed point x^* for λ_0 with $\left|f'_{\lambda_0}(x^*)\right| > 1$, then there is an interval I about x^* and an interval J about λ_0 such that, if $\lambda \in J$, then

(a) f_λ has a unique fixed point which is a source in I, and

(b) all orbits $x_{n+1} = f_\lambda(x_n)$ with starting point $x_0 \in I$ and $x_0 \neq x^*$ eventually leave the interval I.